

Moving horizon \mathcal{H}_∞ -control of constrained periodically time-varying systems^{*}

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Abstract: The paper presents a moving horizon \mathcal{H}_∞ -control approach for the class of linear periodically time-varying systems. By solving repeatedly online a semi-definite program subject to linear matrix inequality constraints, the ℓ_2 gain from the energy bounded external disturbance to the performance output is minimized at each sampling instant. The resulting feedback control strategy guarantees satisfaction of state and input constraints. A numerical example illustrates the effectiveness of the proposed moving horizon \mathcal{H}_∞ -controller.

Keywords: Predictive control, robust control, periodic systems, linear matrix inequalities

1. INTRODUCTION

Model predictive control (MPC), receding horizon control (RHC), or moving horizon control (MHC) is an optimization based control method. In its general form a control sequence is determined by optimizing a finite horizon cost function at each sampling instant, based on an explicit model of the system and the current state measurement. The first part of the obtained control input is applied to the system. At the next sampling instant, the optimization problem is solved again based on new measurements, and the control input is updated. Due to its ability to explicitly handle state and input constraints, MHC has received much interest in both academic community and industrial applications over the last decades, see, for example, Mayne et al. (2000) and Qin and Badgwell (2003).

Periodically time-varying systems (or shortly, *periodic systems*) are of great importance for both control theory and applications. Some well-known real-life problems that involve control of periodic systems are magnetic satellite control problems, see, for example, Wisniewski (1996) and Psiaki (2001), and control of helicopters, see Arcara et al. (2000). Furthermore, as it was, for example, recently pointed out in Gondhalekar and Jones (2009), time-invariant systems that are controlled by asynchronous inputs can be modeled by periodic systems. An excellent overview on existing results for both analysis and controller synthesis of linear periodic systems is provided by the recent monograph Bittanti and Colaneri (2009).

An increasing interest in designing stabilizing MHC schemes for periodic systems could be observed in recent years. For example, the conservativeness of the linear matrix inequalities (LMIs) based MHC approach Kothare et al. (1996) for linear uncertain systems has been reduced

in Böhm et al. (2009a) and Reble et al. (2009) for the class of linear (uncertain) periodic systems. Gondhalekar and Jones (2009) propose the solution of the discrete-time periodic algebraic Riccati equation to derive a suitable terminal cost and terminal set for an MHC scheme that uses a linear periodic model. Similarly, in Böhm et al. (2009b) the terminal cost and ellipsoidal terminal set are obtained by the solution of a semi-definite program (SDP), which makes this approach also applicable to some classes of nonlinear periodic systems. In particular, the MHC approach Reble et al. (2009) is suitable for linear periodic systems with *polytopic uncertain* dynamics, and thus, represents a robustly stabilizing MHC scheme. However, if the system is affected by external disturbances, this approach is not applicable. This motivates the derivation of a moving horizon \mathcal{H}_∞ -controller suitable for periodic systems subject to energy bounded external disturbances, which is based on the concept of ℓ_2 stability, see, for example, Khalil (2002). In particular, the goal of this paper is to extend the results of Chen and Scherer (2006) and Yu et al. (2009) towards designing a novel MHC controller with guaranteed ℓ_2 performance for state and input constrained discrete-time periodic systems. The approach relies on an SDP subject to LMIs, which is solved repeatedly online. The LMI conditions are derived from the dissipation inequality used, for example, in Chen and Scherer (2006), Yu et al. (2009), and Khalil (2002). By solving the SDP at each sampling instant, a feedback matrix is calculated online such that the ℓ_2 gain from the disturbance to the considered performance output is minimized.

The remainder of the paper is organized as follows. Section 2 introduces the considered system class. In Section 3 a preliminary result on \mathcal{H}_∞ -control of periodic systems is discussed. Section 4 provides the main result of the paper, namely a moving horizon \mathcal{H}_∞ -controller for periodic systems with guaranteed satisfaction of state and input constraints. A simulation example in Section 5 illustrates the effectiveness of the moving horizon approach. The paper concludes in Section 6 with a brief summary.

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1.1 Preliminaries

Let \mathbb{R} , \mathbb{R}_+ , \mathbb{Z} and \mathbb{Z}_+ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ we define $\Pi_{\geq c}(\leq c) := \{k \in \Pi \mid k \geq c (k \leq c)\}$, $\mathbb{R}_\Pi := \Pi$ and $\mathbb{Z}_\Pi := \{k \in \mathbb{Z} \mid k \in \Pi\}$. For $N \in \mathbb{Z}_{\geq 1}$, $\Pi^N := \Pi \times \dots \times \Pi$. Let $I(0)$ denote the identity (zero) matrix of suitable dimension. Further, let $\sigma(Q)$ denote the spectrum of the square matrix $Q \in \mathbb{R}^{n \times n}$. For a symmetric matrix $Z \in \mathbb{R}^{n \times n}$ let $Z \succ 0$ ($\succeq 0$) denote that Z is positive definite (semi-definite). Moreover, \star is used to denote the symmetric part of a matrix, i.e., $\begin{bmatrix} a & b^T \\ b & c \end{bmatrix} = \begin{bmatrix} a & \star \\ \star & c \end{bmatrix}$. For a vector $x \in \mathbb{R}^n$, $[x]_i$ denotes the i -th element of x and let $\|\cdot\|$ denote the 2-norm, i.e., $\|x\| := \sqrt{\sum_{i=1}^n |[x]_i|^2}$.

2. PERIODICALLY TIME-VARYING SYSTEMS

We consider the class of linear, periodically time-varying systems which are described by

$$x(k+1) = A(k)x(k) + B_u(k)u(k) + B_w(k)w(k), \quad (1)$$

where $x(k) \in \mathbb{R}^{n_x(k)}$ represents the system states, $u(k) \in \mathbb{R}^{n_u(k)}$ are the control inputs, and $w(k) \in \mathbb{R}^{n_w(k)}$ are the external disturbances.

The system matrices are periodically time-varying with time period $N \in \mathbb{Z}_{\geq 1}$, i.e. $A(k+N) = A(k)$, $B_u(k+N) = B_u(k)$, and $B_w(k+N) = B_w(k)$ for all $k \in \mathbb{Z}_+$. The states, inputs, and disturbances are allowed to be of periodically time-varying dimensions $n_x(k) = n_x(k+N)$, $n_u(k) = n_u(k+N)$, and $n_w(k) = n_w(k+N)$ for all $k \in \mathbb{Z}_+$. Accordingly, the dimensions of the system matrices are time-varying as well, which is illustrated in Table 1.

The consideration of periodic systems is not only motivated by practical problems, such as the attitude control of satellites (Psiaki (2001); Wisniewski (1996)) or control of helicopter rotors (Arcara et al. (2000)), which directly yield models with periodically time-varying dynamics. Often, in real-life problems one has to deal with asynchronous inputs, in particular multirate and/or multiplexed inputs. Although the original system dynamics might be time-invariant, one way to tackle such problems is to use an extended state vector yielding a periodic system with periodically time-varying state, input, and disturbance dimensions. Therefore, the extended periodic system can be used for controller design of the original, possibly time-invariant, system, see Gondhalekar and Jones (2009) for details. Of course, the system description (1) also allows for the more common case of time-invariant state, input, and disturbance dimensions.

Assumption 1. At time $k \in \mathbb{Z}_+$, there exists $\beta \in \mathbb{R}_+$ such that

$$\sum_{i=k}^{\infty} w^T(i)w(i) \leq \beta. \quad (2)$$

For the \mathcal{H}_∞ -controller design we define the performance output

$$y(k) = C(k)x(k) + D_u(k)u(k) + D_w(k)w(k), \quad (3)$$

which is of periodically time-varying dimension $n_y(k) = n_y(k+N) \forall k \in \mathbb{Z}_+$. The dimensions of the periodic output matrices $C(k)$, $D_u(k)$ and $D_w(k)$ are illustrated in Table 1.

Remark 1. In practical applications usually the external disturbances are of time-invariant dimension (i.e., $n_w = n_w(k) \forall k \in \mathbb{Z}_+$). However, for generality, in this paper we do consider time-varying dimensions, since this does not introduce any conservativeness into the problem.

Table 1. Dimensions of system matrices.

Matrix	Dimension	Matrix	Dimension
$A(k)$	$n_x(k+1) \times n_x(k)$	$C(k)$	$n_y(k) \times n_x(k)$
$B_u(k)$	$n_x(k+1) \times n_u(k)$	$D_u(k)$	$n_y(k) \times n_u(k)$
$B_w(k)$	$n_x(k+1) \times n_w(k)$	$D_w(k)$	$n_y(k) \times n_w(k)$
$Y(k)$	$n_u(k) \times n_x(k)$	$K(k)$	$n_u(k) \times n_x(k)$
$X(k)$	$n_x(k) \times n_x(k)$	$P(k)$	$n_x(k) \times n_x(k)$
$c_i(k)$	$1 \times n_x(k)$	$d_i(k)$	$1 \times n_u(k)$

3. \mathcal{H}_∞ -CONTROL OF PERIODIC SYSTEMS

The preliminary control task considered in this section is to determine a linear state feedback law

$$u(k) = K(k)x(k), \quad (4)$$

with periodic feedback matrix $K(k+N) = K(k) \forall k \in \mathbb{Z}_+$. The time-varying dimensions of $K(k)$ are illustrated in Table 1. The feedback matrix shall be calculated such that the dissipation inequality

$$V(k+1, x(k+1)) - V(k, x(k)) - \gamma \|w(k)\|^2 + \gamma^{-1} \|y(k)\|^2 \leq 0 \quad (5)$$

is satisfied for all $k \in \mathbb{Z}_+$. Furthermore, we require that at each time instant the state and input constraints

$$[x(k), u(k)] \in \mathcal{C}(k) \subset \mathbb{R}^{n_x(k)+n_u(k)} \quad (6)$$

are satisfied. In this paper we assume polytopic constraints

$$\mathcal{C}(k) := \{[\xi, \eta] \in \mathbb{R}^{n_x(k)+n_u(k)} \mid c_i(k)\xi + d_i(k)\eta \leq 1\}, \quad (7)$$

$i \in \mathbb{Z}_{[1,p]}$, where $p \in \mathbb{Z}_+$ is the number of constraints. The feedback law (4) renders the control inputs state dependent, and we obtain the constraint set

$$\mathbb{X}(k) := \{\xi \in \mathbb{R}^{n_x(k)} \mid (c_i(k) + d_i(k)K(k))\xi \leq 1\}, \quad (8)$$

$i \in \mathbb{Z}_{[1,p]}$. Thus, if the state $x(k)$ lies in the constraint set $\mathbb{X}(k)$ at time $k \in \mathbb{Z}_+$, then both state and input constraints (6) are satisfied at this time instant.

In the following, we chose the quadratic storage function

$$V(k, x) := x^T P(k)x. \quad (9)$$

See Table 1 for the dimensions of the time-varying matrix $P(k)$, which is positive definite for all $k \in \mathbb{Z}_+$. With the choice of the quadratic storage function (9) the remaining task is to calculate time-varying matrices $P(k)$ and $K(k)$ such that the dissipation inequality (5) and state and input constraints (8) are satisfied for all $k \in \mathbb{Z}_+$. To solve this problem, we choose a periodic matrix $P(k+N) = P(k) \forall k \in \mathbb{Z}_+$. As in the periodic Lyapunov lemma, see, for example, Bittanti and Colaneri (2009),

this ansatz reduces the complexity of the given problem to a finite number of conditions which can be stated as LMIs. The basic idea is to calculate time-varying, robustly periodic invariant ellipsoids defined by the matrix $P(k)$, which are contained in the constraint polytope (8) at the corresponding time instant $k \in \mathbb{Z}_+$. For this, we use the following lemma (see, for example, Boyd et al. (1994)).

Lemma 1. (*Ellipsoid contained in polytope*) Let $0 \prec E \in \mathbb{R}^{n \times n}$, $\mu \in \mathbb{R}_+$, and $w_i \in \mathbb{R}^{1 \times n}$, $\forall i \in \mathbb{Z}_{[1,p]}$ with $p \in \mathbb{Z}_+$. The ellipsoid $\mathcal{E} := \{y \in \mathbb{R}^n | y^\top E y \leq \mu\}$ is contained in the polytope $\mathcal{C} := \{y \in \mathbb{R}^n | w_i y \leq 1, i \in \mathbb{Z}_{[1,p]}\}$, if and only if

$$w_i(\mu P^{-1})w_i^\top \leq 1, \forall i \in \mathbb{Z}_{[1,p]}. \quad (10)$$

The following theorem states LMI conditions for the calculation of $P(k)$ and $K(k)$ such that the dissipation inequality (5) and state and input constraints are satisfied for all $k \in \mathbb{Z}_+$.

Theorem 1. (*\mathcal{H}_∞ -control of periodic systems*) Let Assumption 1 be satisfied for $k = 0$. Let $X(k) \in \mathbb{R}^{n_x(k) \times n_x(k)}$, $X(k) \succ 0$, $Y(k) \in \mathbb{R}^{n_u(k) \times n_x(k)}$, $k \in \mathbb{Z}_{[0,N-1]}$, and $\gamma \in \mathbb{R}_+$ be a feasible solution to

$$\begin{bmatrix} X(k) & \star & \star & \star \\ 0 & \gamma I & \star & \star \\ A(k)X(k) + B_u(k)Y(k) & B_w(k) & X(k+1) & \star \\ C(k)X(k) + D_u(k)Y(k) & D_w(k) & 0 & \gamma I \end{bmatrix} \succeq 0, \quad (11a)$$

$$\begin{bmatrix} X(k)c_i^\top(k) + Y^\top(k)d_j^\top(k) & \star \end{bmatrix} \succeq 0, \quad (11b)$$

for all $i \in \mathbb{Z}_{[1,p]}$ and $k \in \mathbb{Z}_{[0,N-1]}$, where $X(N) := X(0)$. Let $P(k) := X^{-1}(k)$, $K(k) := Y(k)X^{-1}(k)$ for all $k \in \mathbb{Z}_{[0,N-1]}$, and $P(k+N) := P(N)$ and $K(k+N) := K(k)$ for all $k \in \mathbb{Z}_+$. Then, with the state feedback law (4) the following holds:

- (i) The ℓ_2 gain from the disturbance $w(k)$ to the performance output $y(k)$ is less than (or equal to) γ .
- (ii) If the initial state $x(0)$ satisfies

$$\gamma\beta + x^\top(0)P(0)x(0) \leq \alpha, \quad (12)$$

where $\alpha \in \mathbb{R}_+$, then

- a. all perturbed state trajectories are such that at time $k \in \mathbb{Z}_+$ the state $x(k)$ lies in the ellipsoid

$$\mathbb{E}(k) := \{x \in \mathbb{R}^{n_x(k)} | x^\top P(k)x \leq \alpha\}, \quad (13)$$

- b. the constraints (8) are satisfied for all $k \in \mathbb{Z}_+$.

Proof 1. *Part (i):* We have to show that the ℓ_2 gain from the disturbance $w(k)$ to the performance output $y(k)$ is bounded by γ , which is according to Lin and Byrnes (1994, 1996) the case if the dissipation inequality (5) holds for all $k \in \mathbb{Z}_+$. Substituting $X(k)$ and $Y(k)$ in (11a) as defined in the theorem by $P(k)$ and $K(k)$, $k \in \mathbb{Z}_{[0,N-1]}$, we have

$$\begin{bmatrix} P^{-1}(k) & \star & \star & \star \\ 0 & \gamma I & \star & \star \\ (A(k) + B_u(k)K(k))P^{-1}(k) & B_w(k) & P^{-1}(k+1) & \star \\ (C(k) + D_u(k)K(k))P^{-1}(k) & D_w(k) & 0 & \gamma I \end{bmatrix} \succeq 0,$$

$k \in \mathbb{Z}_{[0,N-1]}$. By pre- and post-multiplying with the matrix $\text{diag}(P(k), I, I, I)$, applying the Schur complement, exploiting the periodicity of the occurring matrices, pre- and post-multiplying with $x(k)$ and its transposed, and

using the system dynamics (1), we conclude that the dissipation inequality (5) is satisfied for all $k \in \mathbb{Z}_+$.

Part (ii-a): Consider the storage function (9). Summing up inequality (5) and using Assumption 1 yields

$$V(k, x(k)) + \gamma^{-1} \sum_{i=0}^{k-1} \|y(i)\|^2 \leq V(0, x(0)) + \gamma\beta \quad (14)$$

for all $k \in \mathbb{Z}_+$. Since $\|y(k)\|^2 \geq 0 \forall k \in \mathbb{Z}_+$, we conclude

$$x^\top(k)P(k)x(k) \leq x^\top(0)P(0)x(0) + \gamma\beta, \forall k \in \mathbb{Z}_+. \quad (15)$$

Thus, if (12) is satisfied, then $x(k) \in \mathbb{E}(k)$ for all $k \in \mathbb{Z}_+$.

Part (ii-b): Since the ellipsoid $\mathbb{E}(k)$ contains the state $x(k)$ at time $k \in \mathbb{Z}_+$, state and input constraints are satisfied for all $k \in \mathbb{Z}_+$ if we can show that $\mathbb{E}(k)$ is contained in the constraint set $\mathbb{X}(k)$ at the corresponding time instant. Substituting $X(k)$ and $Y(k)$ in (11b) as defined in the theorem by $P(k)$ and $K(k)$, and applying the Schur complement yields

$$(c_i(k) + d_i(k)K(k))(P^{-1}(k)\alpha)(c_i(k) + d_i(k)K(k))^\top \leq 1$$

for all $i \in \mathbb{Z}_{[1,p]}$ and (due to the periodicity of the occurring matrices and vectors) for all $k \in \mathbb{Z}_+$. Thus, we conclude from Lemma 1 that $\mathbb{E}(k) \subset \mathbb{X}(k)$ for all $k \in \mathbb{Z}_+$. ■

The drawback of the controller design procedure according to Theorem 1 is its conservativeness. The derived controller has to deal with the trade-off between constraint satisfaction and disturbance rejection. This often may lead to conservative feedback matrices $K(k)$ and poor control performance. If the \mathcal{H}_∞ -controller is desired to reject rather large disturbances, one has to choose a large value of α . However, for a large α , condition (11b) restricts the sets of feasible matrices X and Y . This often leads to low control performance even if the actual disturbance is very small. To overcome this problem, in the following section we extend the results obtained so far and introduce an \mathcal{H}_∞ -control approach in a moving horizon fashion, similar to the results proposed by Yu et al. (2009) for polytopic uncertain linear systems and Chen and Scherer (2006) for time-invariant linear systems. The key idea is to formulate a semi-definite program which is solved repeatedly online based on current state measurements. Thus, the feedback law is adjusted online and therefore allows to online deal with the trade-off between disturbance rejection and constraint satisfaction.

4. MOVING HORIZON \mathcal{H}_∞ -CONTROL

The conservativeness of the control approach derived in the previous section motivates the derivation of an \mathcal{H}_∞ -controller in a moving horizon fashion, i.e. the feedback matrices are calculated repeatedly online based on current measurements of the system states. For this, we associate matrices, states, and inputs, which are predicted at time $k \in \mathbb{Z}_+$ for the future time $k+j \in \mathbb{Z}_{\geq k}$, $j \in \mathbb{Z}_+$, by the time index $k+j|k$.

The proposed moving horizon \mathcal{H}_∞ -controller relies on the following semi-definite program which is solved repeatedly at each time instant $k \in \mathbb{Z}_+$.

Problem 1. At time $k \in \mathbb{Z}_+$, let $x(k)$ be the current system state, and solve the optimization problem

$$\underset{X(k+j|k), Y(k+j|k), \gamma(k)}{\text{minimize}} \gamma(k) \quad (16a)$$

subject to

$$\begin{bmatrix} \alpha - \gamma(k)\beta & \star \\ x(k) & X(k|k) \end{bmatrix} \succeq 0, \quad (16b)$$

$$\begin{bmatrix} X(k+j|k) & \star & \star & \star \\ 0 & \gamma(k)I & \star & \star \\ \Delta(k+j) & B_w(k+j) & X(k+j+1|k) & \star \\ \Gamma(k+j) & D_w(k+j) & 0 & \gamma(k)I \end{bmatrix} \succeq 0, \quad (16c)$$

$$\begin{bmatrix} \frac{1}{\alpha} & \star \\ X(k+j|k)c_i^T(k+j) & X(k+j|k) \\ +Y^T(k+j|k)d_j^T(k+j) & \end{bmatrix} \succeq 0, \quad (16d)$$

$$X(k|k) - X(k|k-1) \succeq 0, \quad (16e)$$

for all $j \in \mathbb{Z}_{[0, N-1]}$ and all $i \in \mathbb{Z}_{[1, p]}$, with $X(k+N|k) := X(k|k)$, where (16e) is omitted for $k=0$, and

$$\Delta(k+j) := A(k+j)X(k+j) + B_u(k+j)Y(k+j),$$

$$\Gamma(k+j) := C(k+j)X(k+j) + D_u(k+j)Y(k+j).$$

The proposed moving horizon \mathcal{H}_∞ -control strategy is given by the following algorithm.

Algorithm 1. At each time instant $k \in \mathbb{Z}_+$, measure the state $x(k)$, solve the optimization problem (16), and apply the control input $u(k) := K(k)x(k)$, where $K(k) := Y(k|k)X^{-1}(k|k)$.

The following theorem discusses the properties of the controller defined in Algorithm 1.

Theorem 2. (*Moving horizon \mathcal{H}_∞ -control*) Let Assumption 1 be satisfied. Suppose that Problem 1 is feasible for all $k \in \mathbb{Z}_+$. Then, with $P(k) := X^{-1}(k|k) \forall k \in \mathbb{Z}_+$, the moving horizon \mathcal{H}_∞ -control strategy according to Algorithm 1 guarantees that the following is satisfied:

(i) The dissipation inequality

$$x^T(0)P(0)x(0) \geq \sum_{i=0}^k \gamma^{-1}\|y(i)\|^2 - \gamma\|w(i)\|^2 \quad (17)$$

holds for all $k \in \mathbb{Z}_+$, where $\gamma := \max\{\gamma(j)\}_{j \in \mathbb{Z}_{[0, k]}}$.

(ii) The ℓ_2 -gain from the disturbance $w(k)$ to the output $y(k)$ is less than (or equal to) γ .

(iii) The constraints (8) are satisfied for all $k \in \mathbb{Z}_+$.

Proof 2. Clearly, any solution to (16) satisfies the conditions of Theorem 1. Therefore, from satisfaction of (5) we know that

$$\begin{aligned} x^T(k)P(k)x(k) - x^T(k+1)P(k+1)x(k+1) \\ \geq \gamma^{-1}(k)\|y(k)\|^2 - \gamma(k)\|w(k)\|^2 \end{aligned} \quad (18)$$

holds for all $k \in \mathbb{Z}_+$, where $P(k+1|k) := X^{-1}(k+1|k)$ for all $k \in \mathbb{Z}_+$. Due to condition (16e), which is satisfied for all $k \in \mathbb{Z}_{\geq 1}$, we have $P(k+1|k) \succeq P(k+1)$ for all $k \in \mathbb{Z}_+$, and therefore,

$$\begin{aligned} x^T(k)P(k)x(k) - x^T(k+1)P(k+1)x(k+1) \\ \geq \gamma^{-1}(k)\|y(k)\|^2 - \gamma(k)\|w(k)\|^2 \end{aligned} \quad (19)$$

for all $k \in \mathbb{Z}_+$. Summing up yields

$$\begin{aligned} x^T(0)P(0)x(0) - x^T(k)P(k)x(k) \\ \geq \sum_{i=0}^k \gamma^{-1}(i)\|y(i)\|^2 - \gamma(i)\|w(i)\|^2. \end{aligned} \quad (20)$$

With $\gamma = \max\{\gamma(i)\}_{i \in \mathbb{Z}_{[0, k]}}$, we have that $\gamma \geq \gamma(i)$ and $\gamma^{-1} \leq \gamma(i)$ for all $i \in \mathbb{Z}_{[0, k]}$. Thus,

$$\begin{aligned} x^T(0)P(0)x(0) \geq \sum_{i=0}^k \gamma^{-1}\|y(i)\|^2 - \gamma\|w(i)\|^2 \\ + x^T(k)P(k)x(k). \end{aligned} \quad (21)$$

Since $P(k) \succ 0 \forall k \in \mathbb{Z}_+$, statements (i) and (ii) follow.

According to the proof of Theorem 1, condition (16d) implies that $\mathbb{E}(k) \subset \mathbb{X}(k)$ for all $k \in \mathbb{Z}_+$, where $\mathbb{E}(k)$ is defined as in (13). Applying the Schur complement to (16b) yields

$$x^T(k)P(k)x(k) \leq \alpha - \gamma(k)\beta. \quad (22)$$

Thus, $x(k) \in \mathbb{E}(k)$. Since Assumption 1 holds, it follows from (19) that

$$x^T(k+1)P(k+1)x(k+1) \leq x^T(k)P(k)x(k) + \gamma(k)\beta$$

for all $k \in \mathbb{Z}_+$. Hence,

$$x^T(k+1)P(k+1)x(k+1) \leq \alpha \quad \forall k \in \mathbb{Z}_+, \quad (23)$$

which implies $x(k+1) \in \mathbb{E}(k+1) \forall k \in \mathbb{Z}_+$. We finally conclude that $x(k) \in \mathbb{E}(k) \subset \mathbb{X}(k)$ for all $k \in \mathbb{Z}_+$, which proves statement (iii). ■

The proposed controller allows to online adjust the feedback matrices at each time instant. In the case of small disturbances, this allows to improve controller performance, whereas in the presence of rather large disturbances constraint satisfaction is achieved by paying the price of lower performance, i.e., the moving horizon \mathcal{H}_∞ -controller allows to online deal with the trade-off between performance and constraint satisfaction.

To derive our result, we made use of Assumption 1, which was required to be satisfied at each time instant. This means, that at each time instant we assumed the future disturbance energy to be bounded by β . However, in the case of rather large, unforeseen disturbances, Assumption 1 might be violated and feasibility of Problem 1 might be lost. As discussed in Chen and Scherer (2006), this makes it necessary to be prepared to switch to alternative control strategies if the system is affected by rather larger disturbances. Alternatively, one could pay the price of violating the constraints, which is however often not possible or acceptable in practical control problems. At this point, it is important to notice that any \mathcal{H}_∞ -controller which aims at guaranteeing satisfaction of state and input constraints at all time instants, requires the assumption of the disturbance energy to be bounded.

In the following section a numerical example illustrates the effectiveness of the proposed moving horizon \mathcal{H}_∞ -control scheme.

Remark 2. Condition (16e) is required since we consider periodic systems with time-varying dimensions. In the time-invariant case, condition (16e) can be replaced by the less conservative condition (11c) in Chen and Scherer (2006) or, alternatively, by (24c) in Yu et al. (2009).

5. SIMULATION RESULTS

The moving horizon \mathcal{H}_∞ controller is applied to a two-dimensional example system with period $N = 2$ given by the matrices

$$A(0) := \begin{bmatrix} 0.9 & 0 \\ 0 & 0.8 \end{bmatrix}, \quad A(1) := \begin{bmatrix} 1.2 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}, \quad (24a)$$

$$B_u(0) := \begin{bmatrix} 0.3 \\ 1.5 \end{bmatrix}, \quad B_u(1) := \begin{bmatrix} 1.8 \\ 1.1 \end{bmatrix}, \quad (24b)$$

$$B_w(0) := \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}, \quad B_w(1) := \begin{bmatrix} 0.8 \\ 0.4 \end{bmatrix}, \quad (24c)$$

$$C(0) := [1 \ 0], \quad C(1) := [1 \ 0], \quad (24d)$$

$$D_u(0) = D_u(1) = D_w(0) = D_w(1) := 0. \quad (24e)$$

The example system is subject to hard input constraints

$$-1 \leq u(k) \leq 1, \quad \forall k \in \mathbb{Z}_+. \quad (25)$$

Thus, $p = 1$ and $c_1(0) = c_1(1) := [0 \ 0]$ and $d_1(0) = d_1(1) := 1$. The eigenvalues of the monodromy matrices $\Phi(0) := A(1)A(0)$ and $\Phi(1) := A(0)A(1)$ are $\sigma(\Phi(0)) = \sigma(\Phi(1)) = [1.1856 \ 0.5344]$. Hence, the considered example system is unstable, see, for example, Bittanti and Colaneri (2009) for stability analysis of periodic systems.

For the simulation with the proposed moving horizon \mathcal{H}_∞ -controller we have chosen $\alpha = 300$, $\beta = 15$, and the initial condition $x(0) = [0 \ 0]^T$. The system is affected by the disturbance signal $w(k) = \sqrt{3}$ for all $k \in \mathbb{Z}_{[5,9]}$ and $w(k) = 0$ for all $k \in \mathbb{Z}_{[0,4]}$ and all $k \in \mathbb{Z}_{>9}$, yielding $\sum_{i=0}^{\infty} \|w(i)\|^2 = \beta$. The obtained simulation results are depicted in Figure 1. The controller reacts on the disturbance and steers the system back to the origin without violating the input constraints. The reaction on the disturbance signal is also nicely depicted by the plot of the evolution of the storage function.

Current research investigates the effect of the novel control approach on the magnetic satellite attitude control problem described in Psiaki (2001), and its performance with respect to existing alternative control schemes for the considered satellite attitude control problem.

6. CONCLUSIONS

We presented a moving horizon \mathcal{H}_∞ -control scheme for linear periodically time-varying systems subject to external disturbances. The approach is based on the repeated online solution of a semi-definite program subject to LMI conditions. At each time instant, a feedback matrix is calculated such that the associated control law minimizes the ℓ_2 gain from the energy bounded disturbance to the performance output, while state and input constraints are satisfied. A simulation example illustrated the effectiveness of the proposed controller.

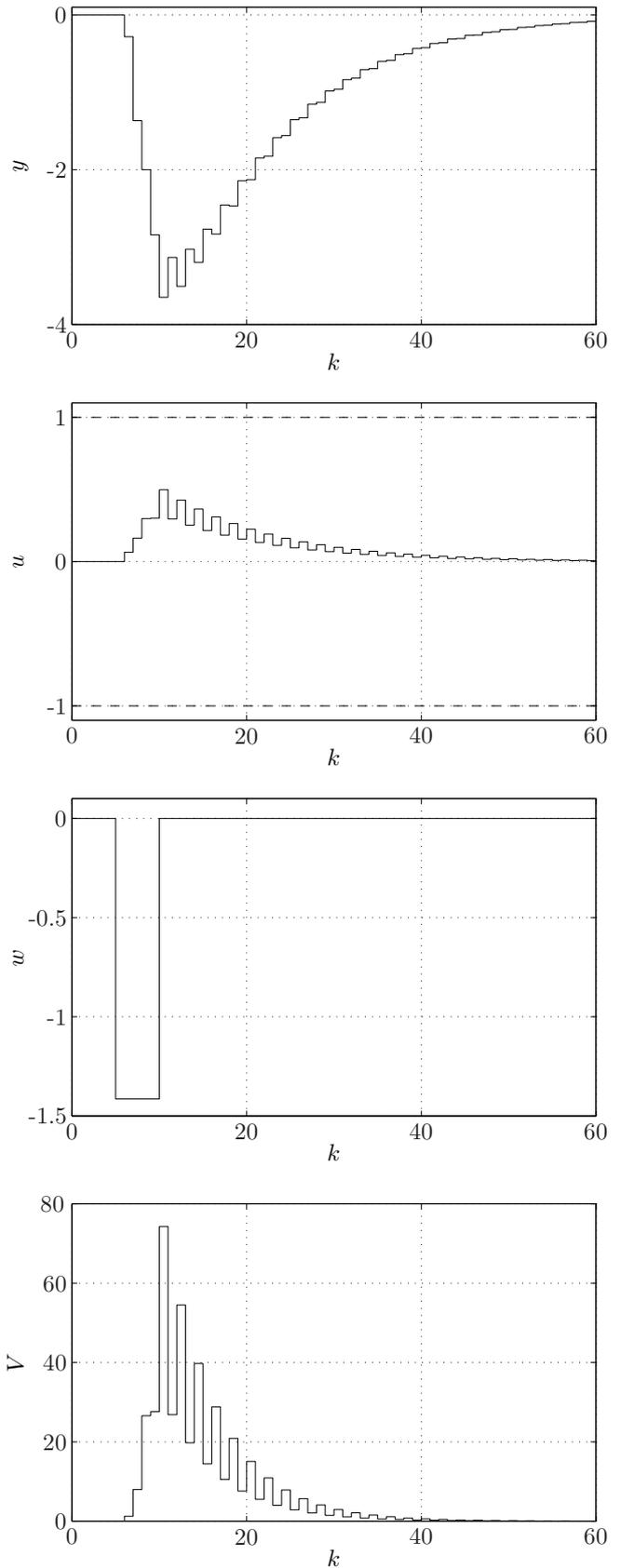


Fig. 1. Plots of performance output y , control input u , disturbance signal w , and storage function V .

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